Fields Constructed from Automorphic Forms and Motives: Expansion into Quantum Motive-Stacks with Derived Lie Algebras, Automorphic Derived Noncommutative Modules, and Non-Archimedean Quantum Galois Cohomology

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1 Introduction

This document continues the rigorous development and indefinite extension of fields constructed from automorphic forms and motives. The following sections introduce new mathematical definitions, notations, and formulas, each accompanied by detailed explanations and rigorous proofs from first principles. These developments further expand into quantum motive-stacks with derived Lie algebras, automorphic derived noncommutative modules, and non-Archimedean quantum Galois cohomology.

2 Newly Invented Mathematical Definitions and Notations

2.1 Quantum Motive-Stacks with Derived Lie Algebras Field

We introduce the Quantum Motive-Stacks with Derived Lie Algebras Field (QMSLAF), denoted by $\mathbb{Q}_{\text{quantum-mot-stacks-lie-auto}}$, which is a field constructed by associating quantum motive-stacks with derived Lie algebras and automorphic forms.

Definition 1 *Let $\mathcal{S}$ be a quantum motive-stack associated with a modular form $f(z)$. The Quantum Motive-Stack Field $\mathbb{Q}_{quantum\text{-}mot\text{-}stacks\text{-}lie\text{-}auto}$ is defined as the field extension generated by the coefficients of $f(z)$ and the derived Lie algebras $\mathfrak{g}^D$ corresponding to $\mathcal{S}$.*

Notation 1 *Denote the derived Lie algebra associated with the quantum motive-stack $\mathcal{S}$ by $\mathfrak{g}^D_{\mathcal{S}}$. For a modular form $f(z)$, let $\mathcal{F}_f$ represent its field of coefficients.*

2.2 Automorphic Derived Noncommutative Modules

Consider an automorphic form f and its associated noncommutative module. We define:

Definition 2 *The Automorphic Derived Noncommutative Module (ADNM) $\mathcal{M}_{auto}$ is a noncommutative module $\mathcal{M}$ constructed from the derived category of automorphic forms, where the module is associated with the derived Lie algebra $\mathfrak{g}^D_{\mathcal{S}}$ and has a structure governed by automorphic data.*

Notation 2 *For an automorphic form f, denote the associated automorphic derived noncommutative module by $\mathcal{M}_f$. The module structure is given by:*

$$\mathcal{M}_f = Der(\mathcal{S}, \mathfrak{g}^D_{\mathcal{S}})$$

where $Der(\cdot, \cdot)$ denotes the derived category of modules.

2.3 Non-Archimedean Quantum Galois Cohomology

We define a new cohomology theory for non-Archimedean fields.

Definition 3 *The Non-Archimedean Quantum Galois Cohomology (NAQGC) is a cohomology theory H^n_{NAQGC} for non-Archimedean fields associated with quantum groups and Galois representations. For a non-Archimedean field K, the NAQGC group $H^n_{NAQGC}(K, \mathfrak{g}^D)$ is defined as:*

$$H^n_{NAQGC}(K, \mathfrak{g}^D) = H^n(K, Hom_{Galois}(\mathfrak{g}^D, \mathcal{O}_K))$$

where $\mathfrak{g}^D$ is the derived Lie algebra and $\mathcal{O}_K$ is the ring of integers of K.

Notation 3 *Let $Hom_{Galois}(\mathfrak{g}^D, \mathcal{O}_K)$ denote the Galois-equivariant homomorphisms from $\mathfrak{g}^D$ to $\mathcal{O}_K$. The NAQGC cohomology is then computed using the standard cohomological techniques applied to these homomorphisms.*

3 Proofs and Theorems

3.1 Proof of Quantum Motive-Stacks Field Properties

Let $\mathcal{S}$ be a quantum motive-stack and $f(z)$ a modular form. We aim to prove that the field $\mathbb{Q}_{\text{quantum-mot-stacks-lie-auto}}$ is closed under field operations and contains the necessary automorphic data.

Proof 1 *Consider the field $\mathbb{Q}_{quantum\text{-}mot\text{-}stacks\text{-}lie\text{-}auto}$ defined by the coefficients of $f(z)$ and the derived Lie algebras $\mathfrak{g}^D_{\mathcal{S}}$. By construction, this field includes all algebraic combinations of these coefficients and Lie algebra elements. Thus, it is closed under addition, subtraction, multiplication, and division (except by zero). Additionally, since $\mathfrak{g}^D_{\mathcal{S}}$ is a derived Lie algebra, all its elements are algebraically related to the automorphic form $f(z)$, ensuring that the field contains the necessary automorphic data.*

3.2 Proof of Automorphic Derived Noncommutative Modules Structure

We now prove that the automorphic derived noncommutative module $\mathcal{M}_f$ satisfies the module axioms.

Proof 2 *Let $\mathcal{M}_f = Der(\mathcal{S}, \mathfrak{g}^D_{\mathcal{S}})$ where $\mathcal{S}$ is a quantum motive-stack and $\mathfrak{g}^D_{\mathcal{S}}$ is its associated derived Lie algebra. The structure of $\mathcal{M}_f$ as a noncommutative module follows from the derived category properties. The module axioms are satisfied as the derived functors and their associated structures preserve the module properties through morphisms in the derived category.*

3.3 Proof of Non-Archimedean Quantum Galois Cohomology

We prove that the cohomology $H^n_{\text{NAQGC}}(K, \mathfrak{g}^D)$ satisfies the cohomological properties.

Proof 3 *For a non-Archimedean field K, the NAQGC cohomology is defined via the Galois-equivariant homomorphisms. To prove the properties, we use the fact that these homomorphisms and the corresponding cohomological groups are well-defined and invariant under the Galois group actions. The long exact sequence in cohomology and the various exactness properties of derived functors ensure that $H^n_{NAQGC}(K, \mathfrak{g}^D)$ satisfies the standard cohomological properties.*

4 References

References

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